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A geometric construction of iterative formulas of order three

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ABSTRACT

In this paper, we consider a geometric construction for improving the order of convergence of iterative formulas of order two. Using this approach, new third-order modifications of Newton's method are derived. A comparison with other existing methods is given.

1. Introduction

This paper is concerned with iterative methods to find a simple root α, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, of a nonlinear equation $f(x) = 0$ that uses no higher than the second derivative of f.

The best known iterative method for the calculation of α is Newton's method defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

where x_0 is an initial approximation sufficiently close to α. This method is quadratically convergent [1].

There exists a modification of Newton's method with third-order convergence due to Potra and Pták [2] defined by

$$x_{n+1} = x_n - \frac{f(x_n) + f'(x_n - f(x_n)/f'(x_n))}{f'(x_n)}. \tag{2}$$

Some Newton-type methods with third-order convergence that do not require the computation of second derivatives have been developed [3–15]. To obtain some of those iterative methods the Adomian decomposition method was applied in [3,4], He's homotopy perturbation method in [5,6] and Liao's homotopy analysis method in [7]. Some of the other methods have been derived by considering different quadrature formulas for the computation of the integral arising from Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)\mathrm{d}t. \tag{3}$$

Weerakoon and Fernando [8] applied the rectangular and trapezoidal rules to the integral of (3) to rederive Newton's method and arrive at the cubically convergent method

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$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n - f(x_n)/f'(x_n))}, \tag{4}$$

while Frontini and Sormani [9] obtained the cubically convergent method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - f(x_n)/(2f'(x_n)))}, \tag{5}$$

by considering the midpoint rule.

In [10], Homeier derived the following cubically convergent iteration scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(x_n - f(x_n)/f'(x_n))}\right), \tag{6}$$

by considering Newton's theorem for the inverse function $x = f(y)$ instead of $y = f(x)$. This scheme has also been derived by Özban in [11] by using arithmetic mean of $f'(x_n)$ and $f'(x_n - f(x_n)/f'(x_n))$ instead of $f'(x_n)$ in Newton's method (1).

Kou et al. in [12] considered Newton's theorem on a new interval of integration and arrived at the following cubically convergent iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n + f(x_n)/f'(x_n)) - f(x_n)}{f'(x_n)}. \tag{7}$$

The above-mentioned methods have order of convergence three but per iteration they require three evaluations for the function f and its first derivatives f', and no evaluations of the second or higher derivatives, which is important and interesting from the practical point of view and becomes active now. In this paper, we develop the third-order modifications of Newton's method which improve the existing second-order methods. To that end we present a detailed description of how to construct iterative methods of order three from iteration functions of order two as well as some illustrations. Finally, the comparison with other third-order methods is given.

2. Iterative methods and convergence analysis

Let x_n be an nth iterate. To develop new methods we consider the tangent line to the curve $y = f(x)$ at the point (x_n, y_n) and also an auxiliary curve defined by the function $h(x) = g(x_n)(x - x_n)$ passing through the point $(x_n, 0)$ where $g : \mathbf{R} \to \mathbf{R}$ is a function to be determined later.

At the intersection point $(x_{n+1}, h(x_{n+1}))$ of the tangent line to the curve $y = f(x)$ at (x_n, y_n) with the auxiliary curve $y = h(x)$, we get

$$f(x_n) + f'(x_n)(x_{n+1} - x_n) = g(x_n)(x_{n+1} - x_n). \tag{8}$$

Eq. (8) can be rewritten as follows

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + \frac{g(x_n)}{f'(x_n)}(x_{n+1} - x_n). \tag{9}$$

By replacing x_{n+1} on the right-hand side of (9) by Newton's iterate, we obtain the new iterative method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + \frac{g(x_n)}{f'(x_n)}(z_n - x_n), \tag{10}$$

where $z_n = x_n - \frac{f(x_n)}{f'(x_n)}$, or

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{g(x_n)f(x_n)}{f'^2(x_n)}. \tag{11}$$

If we compute the error equation for the iteration (11) by the help of Maple, we obtain

$$\begin{aligned} e_{n+1} = & -\frac{g(\alpha)}{f'(\alpha)}e_n + \frac{1}{2}\cdot\frac{f'(\alpha)[f''(\alpha) - 2g'(\alpha)] + 3g(\alpha)f''(\alpha)}{f'^2(\alpha)}e_n^2 \\ & + C(f'(\alpha), f''(\alpha), f^{(3)}(\alpha), g(\alpha), g'(\alpha), g''(\alpha))e_n^3 + O(e_n^4), \end{aligned} \tag{12}$$

where $e_n = x_n - \alpha$.

Thus, for any real valued function g satisfying the conditions

$$g(\alpha) = 0, \qquad g'(\alpha) = \frac{1}{2}f''(\alpha), \tag{13}$$

the iteration (11) yields a third-order modification of Newton's method. If we take $g \equiv 0$, then (11) reduces to Newton's method. Many choices of g may be made to derive iterative methods. Of particular interest among those may be the function g satisfying the differential equation

$$g'(t) = \frac{1}{2} f''(t), \tag{14}$$

subject to the conditions (13). The solution of the differential equation is easily found to be

$$g(x) = \frac{1}{2}\left[f'(x) - f'(\phi(x))\right], \tag{15}$$

where ϕ is any iteration function of second-order, that is, any function satisfying the conditions $\phi(\alpha) = \alpha,\ \phi'(\alpha) = 0$. Thus, any given iteration function ϕ of order two gives rise to a third-order iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\left[f'(x_n) - f'(\phi(x_n))\right]\frac{f(x_n)}{f'^2(x_n)}, \tag{16}$$

or

$$x_{n+1} = x_n - \frac{3}{2}\frac{f(x_n)}{f'(x_n)} + \frac{1}{2}\frac{f(x_n)f'(\phi(x_n))}{f'^2(x_n)}. \tag{17}$$

Thus, we have proved the following theorem:

Theorem 2.1. *Assume that the function f is sufficiently smooth in a neighborhood of its root α, where $f'(\alpha) \neq 0$. Let ϕ be an iteration function of order 2, such that ϕ'' is continuous in a neighborhood of α. Then the iterative method defined by (17) converges cubically to α in a neighborhood of α.*

We now consider some known iteration functions of order two given as follows.

$$\phi_1(x) = x - f(x)/f'(x - f(x)), \tag{18}$$

$$\phi_2(x) = x - f(x)/f'(x), \tag{19}$$

$$\phi_3(x) = x - f(x)/(f(x) + f'(x)), \tag{20}$$

$$\phi_4(x) = x - f(x)f'(x)/(f^2(x) + f'^2(x)). \tag{21}$$

(18) is Stirling's iteration function, (19) Newton's iteration function, (20) the iteration function derived in [16] and (21) in [17].

The application of Theorem 2.1 to iteration functions (18)–(21) yields the new third-order iterative methods

$$x_{n+1} = x_n - \frac{3}{2}\frac{f(x_n)}{f'(x_n)} + \frac{1}{2}\frac{f(x_n)f'(z_{n+1})}{f'^2(x_n)}, \tag{22}$$

where

$$z_{n+1} = \phi_1(x_n), \tag{23}$$

$$z_{n+1} = \phi_2(x_n), \tag{24}$$

$$z_{n+1} = \phi_3(x_n), \tag{25}$$

$$z_{n+1} = \phi_4(x_n), \tag{26}$$

respectively. It should be observed that per iteration the obtained methods use but one evaluation of f and two of f' to carry out iterations.

3. Numerical examples and conclusions

The order of convergence ρ can be approximated using the formula

$$\rho \approx \frac{\ln|(x_{n+1} - \alpha)/(x_n - \alpha)|}{\ln|(x_n - \alpha)/(x_{n-1} - \alpha)|}.$$

All computations were done using MAPLE using 64 digit floating point arithmetics (Digits := 64). We accept an approximate solution rather than the exact root, depending on the precision (ϵ) of the computer. We use the following stopping criteria for computer programs: (i) $|x_{n+1} - x_n| < \epsilon$, (ii) $|f(x_{n+1})| < \epsilon$, and so, when the stopping criterion is satisfied, x_{n+1} is taken as the exact root α computed. For numerical illustrations in this section we used the fixed stopping criterion $\epsilon = 10^{-15}$.

Table 1
Comparison of various cubically convergent iterative methods and Newton's method.

	IT	COC	x_*	$f(x_*)$	δ
$f_1, x_0 = 1.27$					
NM	5	2	1.3652300134140968457608068290	2.70e−41	1.83e−21
WF	4	3	1.3652300134140968457608068290	0.0e−01	3.0e−35
MP	4	3	1.3652300134140968457608068290	0.0e−01	2.60e−36
HM	3	2.96	1.3652300134140968457608068290	−4.45e−48	2.07e−16
KM	4	3	1.3652300134140968457608068290	0.0e−01	1.77e−33
CM1	4	3	1.3652300134140968457608068290	0.0e−01	2.19e−31
CM2	4	3	1.3652300134140968457608068290	0.0e−01	3.06e−27
$f_2, x_0 = 2.0$					
NM	6	2	1.4044916482153412260350868178	−2.26e−32	1.08e−16
WF	5	3	1.4044916482153412260350868178	−2.0e−63	6.02e−42
MP	5	3	1.4044916482153412260350868178	−2.0e−63	7.11e−41
HM	4	3	1.4044916482153412260350868178	−2.0e−63	1.08e−24
KM	5	3	1.4044916482153412260350868178	−2.0e−63	5.29e−31
CM1	5	3	1.4044916482153412260350868178	−2.0e−63	6.59e−34
CM2	5	3	1.4044916482153412260350868178	−2.0e−63	5.24e−27
$f_3, x_0 = 0$					
NM	5	2	0.25753028543986076045536730494	1.56e−49	6.64e−25
WF	4	3	0.25753028543986076045536730494	1.0e−63	1.77e−35
MP	3	2.8	0.25753028543986076045536730494	2.07e−55	2.15e−18
HM	4	3	0.25753028543986076045536730494	1.0e−63	1.58e−37
KM	4	3	0.25753028543986076045536730494	1.0e−63	3.22e−32
CM1	4	3	0.25753028543986076045536730494	−1.0e−63	4.85e−34
CM2	4	3	0.25753028543986076045536730494	1.0e−63	2.90e−33
$f_4, x_0 = 1.2$					
NM	5	2	0.73908513321516064165531208767	−1.90e−35	7.16e−18
WF	4	3	0.73908513321516064165531208767	0.0e−01	1.97e−34
MP	4	3	0.73908513321516064165531208767	0.0e−01	2.72e−27
HM	4	3	0.73908513321516064165531208767	0.0e−01	4.0e−29
KM	4	2.99	0.73908513321516064165531208767	−6.07e−57	2.50e−19
CM1	4	3	0.73908513321516064165531208767	1.0e−64	1.84e−34
CM2	4	2.99	0.73908513321516064165531208767	−4.15e−61	9.55e−21
$f_4, x_0 = 5$					
NM	29	2	0.73908513321516064165531208767	−4.89e−33	1.15e−16
WF	6	3	0.73908513321516064165531208767	0.0e−01	3.55e−38
MP	82	2.99	0.73908513321516064165531208767	1.0e−64	3.06e−25
HM			Divergent		
KM			Divergent		
CM1			Divergent		
CM2	13	2.99	0.73908513321516064165531208767	−1.05e−51	1.3e−17
$f_5, x_0 = 2.4$					
NM	6	2	2	9.87e−33	5.74e−17
WF	5	3	2	0.0e−01	9.29e−40
MP	5	3	2	0.0e−01	5.76e−43
HM	4	3	2	3.48e−61	8.87e−21
KM	5	3	2	0.0e−01	2.17e−38
CM1	5	3	2	0.0e−01	8.89e−33
CM2	5	3	2	0.0e−01	2.36e−28
$f_6, x_0 = 2.3$					
NM	6	2	1.8954942670339809471440357381	−2.45e−48	2.28e−24
WF	4	2.99	1.8954942670339809471440357381	−3.0e−64	1.13e−21
MP	4	2.99	1.8954942670339809471440357381	−1.39e−59	3.64e−20
HM	4	3	1.8954942670339809471440357381	−3.0e−64	2.22e−38
KM	4	2.99	1.8954942670339809471440357381	−3.7e−46	8.27e−16
CM1	4	2.99	1.8954942670339809471440357381	−2.50e−53	3.63e−18
CM2	5	3	1.8954942670339809471440357381	−3.0e−64	1.33e−42
$f_6, x_0 = 13$					
NM			Divergent		
WF	6	3	1.8954942670339809471440357381	1.63e−60	1.87e−20
MP	5	3	1.8954942670339809471440357381	−3.0e−64	2.93e−28
HM			Divergent		

(*continued on next page*)

Table 1 (*continued*)

	IT	COC	x_*	$f(x_*)$	δ
KM			Divergent		
CM1			Divergent		
CM2			Divergent		
$f_7, x_0 = 5$					
NM			Divergent		
WF			Divergent		
MP	23	3	−1.2076478271309189270094167584	−4.0e−63	1.51e−24
HM	318	3	−1.2076478271309189270094167584	−3.58e−49	2.60e−17
KM			Divergent		
CM1			Divergent		
CM2	37	2.99	−1.2076478271309189270094167584	−1.51e−46	1.23e−16

We present some numerical test results for various cubically convergent iterative schemes in Table 1. Compared were Newton's method (NM), the method of Weerakoon and Fernando (4) (WF), the method derived from midpoint rule (5) (MP), the method of Homeier (6) (HM), the method of Kou et al. (7) (KM), and the methods (22) with (24) (CM1) and (25) (CM2), respectively, introduced in the present contribution. We remark that chosen for comparison are only the methods which do not require the computation of second or higher derivatives of the function to carry out iterations. We used the following test functions:

$$f_1(x) = x^3 + 4x^2 - 10,$$
$$f_2(x) = \sin^2 x - x^2 + 1,$$
$$f_3(x) = x^2 - e^x - 3x + 2,$$
$$f_4(x) = \cos x - x,$$
$$f_5(x) = (x-1)^3 - 1,$$
$$f_6(x) = \sin x - x/2,$$
$$f_7(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5.$$

As convergence criterion, it was required that the distance of two consecutive approximations δ for the zero was less than 10^{-15}. Also displayed are the number of iterations to approximate the zero (IT), the computational order of convergence (COC), the approximate zero x_*, and the value $f(x_*)$. Note that the approximate zeroes were displayed only up to the 28th decimal places, so it making all looking the same though they may in fact differ.

The test results in Table 1 show that the computed order of convergence of the presented iterative methods is three, which agree with the theoretical result developed in this paper. It is well known that convergence of iteration formula is guaranteed only when the initial approximation is sufficiently near root. In general, it may be divergent when the initial approximation is far from root as this can be observed in Table 1. It can be observed that for most of the functions we tested, the methods introduced in the present presentation show at least equal performance compared to the other known methods of the same order.

References

[1] A.M. Ostrowski, Solution of Equations in Euclidean and Banach Space, Academic Press, New York, 1973.
[2] F.A. Potra, V. Pták, Nondiscrete induction and iterative processes, in: Research Notes in Mathematics, vol. 103, Pitman, Boston, 1984.
[3] S. Abbasbandy, Improving Newton–Raphson method for nonlinear equations by modified Adomian decomposition method, Appl. Math. Comput. 145 (2003) 887–893.
[4] C. Chun, Iterative methods improving Newton's method by the decompositionm method, Comput. Math. Appl. 50 (2005) 1559–1568.
[5] J.H. He, Newton-like iteration methods for solving algebraic equations, Commun. Nonlinear Sci. Numer. Simul. 3 (2) (1998) 106–109.
[6] S. Abbasbandy, Modified homotopy perturbation method for nonlinear equations and comparison with Adomian decomposition method, Appl. Math. Comput. 172 (2006) 431–438.
[7] C. Chun, Construction of Newton-like iteration methods for solving nonlinear equations, Numer. Math. 104 (3) (2006) 297–315.
[8] S. Weerakoon, G.I. Fernando, A variant of Newton's method with accelerated third-order convergence, Appl. Math. Lett. 17 (2000) 87–93.
[9] M. Frontini, E. Sormani, Some variants of Newton's method with third-order convergence, J. Comput. Appl. Math. 140 (2003) 419–426.
[10] H.H.H. Homeier, On Newton-type methods with cubic convergence, J. Comput. Appl. Math. 176 (2005) 425–432.
[11] A.Y. Özban, Some new variants of Newton's method, Appl. Math. Lett. 17 (2004) 677–682.
[12] J. Kou, Y. Li, X. Wang, A modification of Newton method with third-order convergence, Appl. Math. Comput. 181 (2006) 1106–1111.
[13] J.F. Traub, Iterative Methods for the Solution of Equations, Chelsea Publishing Company, New York, 1977.
[14] M. Dehghan, M. Hajarian, One some cubic iterative formulae without derivatives for solving nonlinear equations, Commun. Numer. Methods Eng. (2009) doi:10.1002/cnm.1328.
[15] I. Pavaloiu, E. Catinas, On a Steffensen–Hermite method of order three, Appl. Math. Comput. (2009) doi:10.1016/j.amc.2009.09.006.
[16] X.-Y. Wu, A new continuation Newton-like method and its deformation, Appl. Math. Comput. 112 (2000) 75–78.
[17] V. Mamta, V.K. Kanwar, S. Kukreja, Singh, On a class of quadratically convergent iteration formulae, Appl. Math. Comput. 166 (3) (2006) 633–637.